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ScienceDirect
Journal of Geometry and Physics 57 (2007) 1099–1113
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On generalization of Birkhoff's theorem

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Received 23 December 2004; received in revised form 4 February 2006; accepted 1 September 2006
Available online 10 October 2006

Abstract

A theorem of G.D. Birkhoff states that a spherically symmetric space–time with vanishing Ricci tensor admits a Killing field beyond those given by the spherical action and thus it is static under an additional condition motivated by the exterior Schwarzschild space–time. Several generalizations of this result were obtained for spherically symmetric space–times and also for such more general ones where an isometric action of a 3-dimensional Lie group with orbits of maximal dimension 2 is given. It seems, however, that a complete account of those space–times where such a generalized theorem holds fails even now. A construction of all those spherically symmetric space–times is presented below where a generalization of Birkhoff's theorem holds.

MSC: 53C50; 53C80; 83C20; 83C40

Keywords: Lorentz geometry; General relativity; Symmetries of space–times; Spherically symmetric space–times

1. Introduction

The first generalization of Birkhoff's theorem [2] was obtained by Cahen and Debever who extended its validity in two senses: firstly, instead of spherically symmetric ones they considered space–times where an isometric action of a 3-dimensional Lie group with non-lightlike orbits of maximal dimension 2 is given; secondly, instead of Ricci flat ones they considered Einstein space–times [4]. Later on Goenner discovered that Birkhoff's theorem extends to even more general ones than the Einstein space–times [7]. Actually, Goenner's result was based on earlier developments: on the classification of space–times by Petrov [12], and by Plebański [13], and on the study of those space–times where an isometric action of a 3-dimensional Lie group with orbits of maximal dimension 2 exist [14]. The results of Goenner were later on completed by Barnes [1].

It is to be noted that there is a presentation of Birkhoff's theorem which concentrates on the fact that a spherically symmetric space–time with vanishing Ricci tensor is locally isometric to the Schwarzschild space–time (see e.g. [8], pp. 369–372). For a detailed account of the classical results concerning Birkhoff's theorem see [16].

Actually, the approach adopted in case of the above mentioned generalizations can be summarized roughly as follows: Let $(M, \langle, \rangle)$ be a space–time and

$$\Phi : G \times M \to M$$

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.09.005

the isometric action of a 3-dimensional Lie group G with non-lightlike orbits having maximal dimension 2. Since Birkhoff's original theorem yields a Killing field which is orthogonal everywhere to the orbits of the spherical action, a generalized Birkhoff theorem has to yield a Killing field which is orthogonal to the orbits of Φ everywhere on M; such a Killing field will be called a Birkhoff field here. But as a Killing field is obtainable by solving the Killing equation in coordinates, sufficient conditions for the solvability of this equation were given in terms of the Einstein tensor of the space–time. Thus Birkhoff fields were obtained, at least locally. Yet, as was pointed out by Goenner, a necessary and sufficient condition for the solvability of the Killing equation in terms of the Einstein tensor alone is not likely to be achievable [7], and it seems now that no further work was done on the subject after this observation [10]. Thus the problem as to a complete account of those space–times where a generalization of Birkhoff's theorem holds remained open.

A new approach to the above problem is presented below. In fact, a recent study of the global geometry of spherically symmetric space–times yields the following: If $(M, \langle , \rangle)$ is a spherically symmetric space–time then there is an open and dense subset $M^* \subset M$, the so called principal part of M, such that under fairly general global conditions

$$M^* = L \times_\varrho \mathbb{S}^2$$

holds, which means that M^* is obtainable as a warped product of a 2-dimensional Lorentz manifold L and of a 2-sphere [17]. A necessary and sufficient condition for the validity of a generalized Birkhoff's theorem in the case of $(M, \langle , \rangle)$ is given in terms of the Lorentzian factor L and of the warping function $\varrho : L \to \mathbb{R}^+$. Thus a construction of all those spherically symmetric space–times is obtained where a generalization of Birkhoff's theorem holds. It is also shown how the above mentioned earlier results are obtainable as simple consequences of the new one. Moreover, such examples of spherically symmetric space–times are given which admit Birkhoff fields but are not covered by the earlier results.

2. A construction of spherically symmetric space–times admitting a Birkhoff field

If $(M, \langle , \rangle)$ is a space–time, i.e. a 4-dimensional time-oriented connected Lorentz manifold, then an isometric action

$$\Phi : \mathrm{SO}(3) \times M \to M$$

is said to be *spherical* if the maximum of the dimension of its orbits is 2; in this case $(M, \langle , \rangle)$ is called a *spherically symmetric space–time* ([8], pp. 369–372, [15], p. 261). Some fundamental results concerning the global geometry of spherically symmetric space–times which were obtained earlier [17], and will be applied below, are summarized in what now follows. A spherical action Φ obviously has no 1-dimensional orbits and the connected components of the set of its 0-dimensional orbits are timelike geodesics which are called the *axes* of the action [17]. The 2-dimensional orbits are spacelike compact submanifolds on which the Lorentz metric $\langle , \rangle$ induces Riemannian metrics of positive constant curvature. By an application of the theory of compact Lie group actions the 2-dimensional orbits can be classified as *principal* and *exceptional* ones ([3], pp. 180–181); the union of the principal ones is a connected open dense subset M^* of M by the *principal orbit type theorem* ([3], pp 179–180), and accordingly M^* is called the *principal part* of M here. The restriction of Φ to the invariant set M^* is obviously spherical. Put $G = \mathrm{SO}(3)$ for convenience, and let $G(z) \subset M$ be the orbit of $z \in M^*$ and $T_z^{\perp} G(z)$ the normal space of the orbit at z. Then a smooth involutive distribution

$$\mathcal{N}_z = T_z^{\perp} G(z), \quad z \in M^*$$

is obtained on the principal part. The maximal integral manifolds of $\mathcal{N}$ are totally geodesic and are called the *leaves* of the spherically symmetric space–time. If $L \subset M^*$ is a leaf then all the isotropy subgroups $G_z \subset G = \mathrm{SO}(3)$, $z \in L$, are the same. For each leaf L there is a unique closed totally geodesic submanifold $P \subset M$ such that $L \subset P$ and L is open and dense in P; the closed totally geodesic submanifold P is called a *transverse submanifold* of the spherical action. If a spherically symmetric space–time has an axis then this is included in all the transverse submanifolds of the space–time. Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ as the unit sphere and let

$$\Psi : \mathrm{SO}(3) \times \mathbb{S}^2 \to \mathbb{S}^2$$

be the action obtained by restricting the canonical action of SO(3) on $\mathbb{R}^3$ to $\mathbb{S}^2$. Furthermore, as usual identify SO(2) with the isotropy subgroup $G_a \subset G = \mathrm{SO}(3)$, where $a = (1, 0, 0) \in \mathbb{S}^2$. Let $z \in M^*$ be fixed so that the identity component of G_z is SO(2); since the identity components of the isotropy subgroups at points of M^* yield a complete conjugacy class of SO(2) in SO(3) this can be achieved. Consider the canonical equivariant diffeomorphisms

$$\theta : \mathbb{S}^2 \to \mathrm{SO}(3)/\mathrm{SO}(2), \qquad \chi_z : G(z) \to G/G_z$$

and also the smooth equivariant covering map

$$\kappa : \mathrm{SO}(3)/\mathrm{SO}(2) \to G/G_z.$$

Then there is a unique smooth covering map ω_z which renders commutative the following diagram:

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{\omega_z} & G(z) \\ \theta \big\downarrow & & \chi_z \big\downarrow \\ \mathrm{SO}(3)/\mathrm{SO}(2) & \xrightarrow{\kappa} & G/G_z. \end{array}$$

The principal orbits of Φ are diffeomorphic to $\mathbb{S}^2$ if and only if the map ω_z is a bijection, and then it is a homothety with a factor $\rho(z) \in \mathbb{R}^+$. Thus in this case a smooth function

$$\rho : M^* \to \mathbb{R}$$

is obtained, which extends smoothly to M and is invariant under the action Φ. Let $(M, \langle , \rangle)$ be a spherically symmetric space–time such that its principal orbits are diffeomorphic to $\mathbb{S}^2$; then

$$\rho : M^* \to \mathbb{R}^+$$

is called its *radial function*. Spherically symmetric space–times can be fairly different from the global topological point of view [5,18]; the ones which are the simplest in this respect are given as follows: A spherically symmetric space–time is called *normal* if it satisfies the following three conditions:

1. M is oriented.
2. The spherical action Φ has no exceptional orbits.
3. The leaves are simply connected non-compact submanifolds.

In this case the principal orbits of Φ, being orientable, are diffeomorphic to $\mathbb{S}^2$ ([3], p. 185.)

The global geometry of normal spherically symmetric space–times admits the simplest approach. If $(M, \langle , \rangle)$ is a normal spherically symmetric space–time and $L \subset M^*$ a leaf, then $\langle , \rangle$ induces a Lorentz metric $\langle , \rangle_L$ on L. Consider also the restricted radial function $\varrho = \rho \lceil L$; then the Lorentzian warped product

$$Q = L \times_\varrho \mathbb{S}^2$$

is a space–time with the time orientation inherited from M. Moreover, the canonical action Ψ of SO(3) on $\mathbb{S}^2$ extends to an isometric action

$$\widetilde{\Psi} : \mathrm{SO}(3) \times (L \times_\varrho \mathbb{S}^2) \to L \times_\varrho \mathbb{S}^2.$$

The fact, which is the starting point for the results presented here, is the existence of an isometry

$$\Xi : L \times_\varrho \mathbb{S}^2 \to M^*$$

which is equivariant with respect to the actions $\widetilde{\Psi}$, Φ [17]. Thus the principal part of a normal spherically symmetric space–time which is likewise spherically symmetric can be obtained as a warped product.

Those Killing fields which are given by Birkhoff's theorem and its generalizations are singled out by the following definition in the case of spherically symmetric space–times.

Definition 1. Let $(M, \langle , \rangle)$ be a spherically symmetric space–time; a smooth vector field $X : M \to TM$ is called a *Birkhoff field* if it satisfies the following two conditions:

(1) X is a non-trivial Killing field.
(2) If $z \in M^*$ and L is the leaf passing through z, then $X(z) \in T_z L$ holds.

A characterization of those spherically symmetric space–times which admit a Birkhoff field is presented subsequently. First some basic concepts and facts concerning warped products are summarized which will be applied in what follows.

Consider a warped product $Q = L \times_{\varrho} \mathbb{S}^2$ where L is a 2-dimensional Lorentz manifold; then its product manifold structure yields the canonical projections

$$\pi_L : Q \to L, \qquad \pi_{\mathbb{S}^2} : Q \to \mathbb{S}^2$$

which in turn give rise to canonical lifts of vector fields: If $U \in \mathcal{T}(L)$ then its *canonical lift* is the unique $\widetilde{U} \in \mathcal{T}(Q)$ such that

$$T\pi_L \circ \widetilde{U} = U \circ \pi_L,$$
$$\widetilde{U}(z,s) \in T_{(z,s)}(L \times \{s\}), \quad (z,s) \in Q.$$

The set of such canonical lifts is a subspace $\mathfrak{L}(L) \subset \mathcal{T}(Q)$. If $V \in \mathcal{T}(\mathbb{S}^2)$, then its *canonical lift* is the unique $\widetilde{V} \in \mathcal{T}(Q)$ such that

$$T\pi_{\mathbb{S}^2} \circ \widetilde{V} = V \circ \pi_{\mathbb{S}^2},$$
$$\widetilde{V}(z,s) \in T_{(z,s)}(\{z\} \times \mathbb{S}^2) \subset T_{(z,s)}Q.$$

The set of such canonical lifts is a subspace $\mathfrak{L}(\mathbb{S}^2) \subset \mathcal{T}(Q)$ ([11], p. 205).

Following the usual terminology, tangent vectors

$$v \in T_{(z,s)}(L \times \{s\}), \quad (z,s) \in Q$$

are called *horizontal*, and the tangent vectors

$$w \in T_{(z,s)}(\{z\} \times \mathbb{S}^2)$$

are called *vertical* ([11], pp. 24–25).

A starting point for the subsequent results is a construction of those spherically symmetric space–times which admit Birkhoff fields given by the following simple proposition.

Proposition 1. *Let $(L, \langle,\rangle_L)$ be a connected time-oriented 2-dimensional Lorentz manifold, $S : L \to TL$ its Killing field, and $\varrho : L \to \mathbb{R}^+$ a smooth function which is a first integral of S. Then the spherically symmetric space–time given by the warped product*

$$Q = L \times_{\varrho} \mathbb{S}^2$$

with the time orientation induced by that of L admits a Birkhoff field $X : Q \to TQ$ which is given by

$$X = \widetilde{S} : Q \to TQ$$

as the canonical lift of the Killing field S of $(L, \langle,\rangle_L)$.

Proof. Since X is the canonical lift of S, it is smooth and its values, being horizontal, are tangent to the leaves. In order to show that X is a Killing field, consider smooth vector fields $\widehat{U}, \widehat{V} \in \mathcal{T}(Q)$. Let $g_L,\ g_{\mathbb{S}^2}, g$ be respectively the metrics of the factor manifolds and of the warped product. Then

$$\begin{aligned}(\mathcal{L}_X g)(\widehat{U}, \widehat{V}) &= X(g(\widehat{U}, \widehat{V})) - g([X, \widehat{U}], \widehat{V}) - g(\widehat{U}, [X, \widehat{V}]) \\ &= X\left(g_L(T\pi_L \widehat{U}, T\pi_L \widehat{V}) \circ \pi_L + (\varrho \circ \pi_L) \cdot g_{\mathbb{S}^2}(T\pi_{\mathbb{S}^2} \widehat{U}, T\pi_{\mathbb{S}^2} \widehat{V}) \circ \pi_{\mathbb{S}^2}\right) \\ &\quad - \left(g_L(T\pi_L [X, \widehat{U}], T\pi_L \widehat{V}) \circ \pi_L + (\varrho \circ \pi_L) \cdot g_{\mathbb{S}^2}(T\pi_{\mathbb{S}^2}[X, \widehat{U}], T\pi_{\mathbb{S}^2} \widehat{V}) \circ \pi_{\mathbb{S}^2}\right) \\ &\quad - \left(g_L(T\pi_L \widehat{U}, T\pi_L [X, \widehat{V}]) \circ \pi_L + (\varrho \circ \pi_L) \cdot g_{\mathbb{S}^2}(T\pi_{\mathbb{S}^2} \widehat{U}, T\pi_{\mathbb{S}^2}[X, \widehat{V}]) \circ \pi_{\mathbb{S}^2}\right)\end{aligned}$$

holds for the Lie derivative of the warped product metric g.

1. If $\widehat{U} = \widetilde{U},\ \widehat{V} = \widetilde{V} \in \mathfrak{L}(L)$ then the above expression reduces to the following one:

$$\begin{aligned}(\mathcal{L}_X g)(\widetilde{U}, \widetilde{V}) &= X((g_L(U, V)) \circ \pi_L) - (g_L([S, U], V) - g_L(U, [S, V])) \circ \pi_L \\ &= (S(g_L(U, V)) - g_L([S, U], V) - g_L(U, [S, V])) \circ \pi_L,\end{aligned}$$

since by a basic result ([11], pp. 24–25) the equality $T\pi_L[X, \widetilde{U}] = T\pi_L\widetilde{[S, U]} = [S, U]$ holds and for the same reason $T\pi_L[X, \widetilde{V}] = [S, V]$ is valid. But then the above expression is equal to 0, since S is a Killing field.

2. If $\widehat{U} = \widetilde{U} \in \mathfrak{L}(L)$, $\widehat{V} = \widetilde{V} \in \mathfrak{L}(\mathbb{S}^2)$, then by the above expression of the Lie derivative of the warped product metric

$$(\mathcal{L}_X g)(\widetilde{U}, \widetilde{V}) = 0$$

is obtained by the above already mentioned facts; namely $[X, \widetilde{U}] = \widetilde{[S, U]}$ and $[X, \widetilde{V}] = [\widetilde{S}, \widetilde{V}] = 0$ by basic properties of canonical lifts ([11], pp. 24–25).

3. If $\widehat{U} = \widetilde{U}$, $\widehat{V} = \widetilde{V} \in \mathfrak{L}(\mathbb{S}^2)$, then the above expression for the Lie derivative of the warped product metric, by the already mentioned basic facts, reduces to the following one:

$$(\mathcal{L}_X g)(\widetilde{U}, \widetilde{V}) = X\left((\varrho \circ \pi_L) \cdot (g_{\mathbb{S}^2}(U, V) \circ \pi_{\mathbb{S}^2})\right).$$

But $X(\varrho \circ \pi_L) = (S\varrho) \circ \pi_L = 0$, since ϱ is a first integral of S. Moreover, $g_{\mathbb{S}^2}(U, V) \circ \pi_{\mathbb{S}^2}$ is constant on the leaves and X is tangent to the leaves; therefore $X(g_{\mathbb{S}^2}(U, V) \circ \pi_{\mathbb{S}^2}) = 0$ holds. Consequently, the above expression is equal to 0.

Since $\mathcal{L}_X g$ is a tensor, and as any tangent vector of $Q = L \times_\varrho \mathbb{S}^2$ is obtainable as a linear combination of values of canonical lifts, the proposition follows. □

The following lemma serves to show that the conditions of the preceding proposition are also necessary for the existence of a Birkhoff field.

Lemma 1. *Let $(L, \langle, \rangle_L)$ be a 2-dimensional time-oriented Lorentz manifold, and $X : Q \to TQ$ be a Birkhoff field of the spherically symmetric space–time $Q = L \times_\varrho \mathbb{S}^2$ and L identified with a leaf of Q. Then $S = X \lceil L$ is a Killing field of L and $S\varrho = 0$ holds.*

Proof. The fact that S is a Killing field of L follows from the fact that $L \subset Q$ is totally geodesic (for a proof of the corresponding general theorem in the not essentially Riemannian case, see e.g. [9], vol. II, pp. 59–60). Let g be the Lorentz metric of Q, then for $U \in \mathcal{T}(\mathbb{S}^2)$ and its canonical lift $\widetilde{U} \in \mathcal{T}(Q)$ the following holds:

$$\begin{aligned} 0 &= (\mathcal{L}_X g)(\widetilde{U}, \widetilde{U}) = X(g(\widetilde{U}, \widetilde{U})) - g([X, \widetilde{U}], \widetilde{U}) - g(\widetilde{U}, [X, \widetilde{U}]) \\ &= X((\varrho \circ \pi_L) g_{\mathbb{S}^2}(U, U)\pi_{\mathbb{S}^2}) \\ &= (X(\varrho \circ \pi_L))\, g_{\mathbb{S}^2} + \varrho \circ \pi_L X(g_{\mathbb{S}^2}(U, U) \circ \pi_{\mathbb{S}^2}) \\ &= (S\varrho) \circ \pi_L(g_{\mathbb{S}^2}(U, U) \circ \pi_{\mathbb{S}^2}), \end{aligned}$$

since $g_{\mathbb{S}^2}(U, U)\pi_{\mathbb{S}^2}$ is constant on the leaves and X is tangent to the leaves. □

In order to show that the conditions of the preceding proposition are also sufficient for the existence of a Birkhoff field in a normal spherically symmetric space–time $(M, \langle, \rangle)$, assume that there is a leaf $L \subset M$, a smooth function $\varrho : L \to \mathbb{R}^+$ and a Killing field $S : L \to TL$ such that $S\varrho = 0$ is valid. Then the canonical lift $X = \widetilde{S} : Q \to TQ$ is a Birkhoff field of the spherically symmetric space–time

$$Q = L \times_\varrho \mathbb{S}^2$$

by the preceding proposition. But then by the identification $M^* = Q$ the canonical lift X is a Birkhoff field also of the spherically symmetric space–time M^*. If $M \neq M^*$ then $M = A \cup M^*$ where A is the axis of the space–time since this is normal [17]. But then X obviously extends to a Birkhoff field $\widehat{X}$ of M on putting $\widehat{X} \lceil A = 0$.

3. A necessary and sufficient condition for the existence of a Birkhoff field in terms of the radial function

The construction of a Birkhoff field of a spherically symmetric space–time given by the preceding Proposition 1 is based on some conditions which concern the geometry of a leaf as a 2-dimensional Lorentz manifold; actually the existence of such a Killing field of this 2-dimensional Lorentz manifold is required which leaves the restricted radial function invariant. In what follows, these geometric conditions are replaced by other, analytic, ones which concern only the restricted radial function. In this way it will be possible to answer the question as to the role of the Ricci tensor of a spherically symmetric space–time in assuring the existence of a Birkhoff field.

Lemma 2. *Let $(L, \langle , \rangle_L)$ be a 2-dimensional Lorentz manifold and $\varrho : L \to \mathbb{R}$ a smooth function. Then*

$$\operatorname{grad} \langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L = 2\nabla^L_{\operatorname{grad} \varrho} \operatorname{grad} \varrho$$

holds for the gradient of the smooth function $\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle$ where ∇^L is the Levi-Cività covariant derivation.

Proof. Let V be a smooth local vector field; then

$$\begin{aligned} V \langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L &= V((\operatorname{grad} \varrho)\varrho) = [V, \operatorname{grad} \varrho]\varrho + \operatorname{grad} \varrho (V\varrho) \\ &= \langle (\nabla^L_V \operatorname{grad} \varrho - \nabla^L_{\operatorname{grad} \varrho} V), \operatorname{grad} \varrho \rangle_L + \langle \nabla^L_{\operatorname{grad} \varrho} V, \operatorname{grad} \varrho \rangle_L + \langle V, \nabla_{\operatorname{grad} \varrho} \operatorname{grad} \varrho \rangle \\ &= \frac{1}{2} V \langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L + \langle V, \nabla^L_{\operatorname{grad} \varrho} \operatorname{grad} \varrho \rangle_L \end{aligned}$$

holds and yields the assertion of the lemma. □

Theorem 1. *Let $(M, \langle , \rangle)$ be a normal spherically symmetric space–time, $L \subset M$ a leaf, $\varrho = \rho \lceil L : L \to \mathbb{R}^+$ the corresponding restricted radial function and $\varrho(L) = (\alpha, \omega)$ where $0 \le \alpha < \omega \le \infty$. Then the space–time admits a Birkhoff field if and only if*

$$\begin{aligned} &\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L = \phi \circ \varrho, \\ &\Delta \varrho = \psi \circ \varrho \end{aligned}$$

holds with smooth functions $\phi, \ \psi : (\alpha, \omega) \to \mathbb{R}$ for the gradient and Laplacian of ϱ calculated in the Lorentz manifold $(L, \langle , \rangle_L)$.

Proof. In order to show that the conditions of the theorem are necessary consider a Birkhoff field $X : M \to TM$ and its restriction $S = X \lceil L$ which is a Killing field of $(L, \langle , \rangle_L)$ as already mentioned above. It will be shown first that the following equalities are valid:

$$\begin{aligned} &S\varrho = \langle S, \operatorname{grad} \varrho \rangle_L = 0, \\ &\mathcal{L}_S \operatorname{grad} \varrho = 0, \\ &S\Delta\varrho = 0. \end{aligned}$$

In order to verify the first equality observe that M^* can be identified with $L \times_\varrho \mathbb{S}^2$ by the equivariant isometry Ξ, and thus the first equality is a simple consequence of the preceding Lemma 1.

In order to verify the second equality observe that the zero set of S is nowhere dense in L since S is a non-trivial Killing field there. Consider now a $z \in L$ such that $S(z) \neq 0$ and a $u \in T_z L - \{0_z\}$. Then there is a smooth local vector field U defined on a neighbourhood of z in L and such that

1. $U(z) = u$ is valid,
2. U is tangent to L,
3. $[S, U] = 0$ holds; in fact. there is a coordinate system on a neighbourhood of z such that U is a base field. Now the following holds:

$$\begin{aligned} \langle u, \mathcal{L}_S \operatorname{grad} \varrho(z) \rangle_L &= (\langle U, \mathcal{L}_S \operatorname{grad} \varrho \rangle_L)(z) \\ &= ((S\langle U, \operatorname{grad} \varrho \rangle_L) - \langle [S, U], \operatorname{grad} \varrho \rangle_L)(z) = S(U\varrho) \\ &= (S(U\varrho) - U(S\varrho))(z) = ([S, U]\varrho)(z) = 0. \end{aligned}$$

Since $u \in T_z L$ is arbitrary, this implies that $\mathcal{L}_S \operatorname{grad} \varrho = 0$ is valid.

In order to verify the third equality put $Y = \operatorname{grad} \varrho$; then

$$\Delta \varrho = \operatorname{div} \operatorname{grad} \varrho = \operatorname{div} Y$$

by a definition of the Laplacian ([11], p. 86). Let now $\upsilon \in \Omega^2(L)$ be the canonical volume form of the Lorentz manifold $(L, \langle, \rangle_L)$. Then by a fundamental property of the divergence ([11], pp. 195–196) the following holds:

$$\begin{aligned}
&\mathcal{L}_Y \upsilon = (\operatorname{div} Y)\upsilon,\\
&\mathcal{L}_S \mathcal{L}_Y \upsilon = (S(\operatorname{div} Y))\upsilon + (\operatorname{div} Y)\mathcal{L}_S \upsilon,\\
&\mathcal{L}_{[S,Y]}\upsilon + \mathcal{L}_Y \mathcal{L}_S \upsilon = (S(\operatorname{div} Y))\upsilon,\\
&0 = (S(\operatorname{div} Y))\upsilon,
\end{aligned}$$

since $\mathcal{L}_S \upsilon = 0$ and $[S, Y] = \mathcal{L}_S \operatorname{grad} \varrho = 0$ is valid. But then $S(\Delta \varrho) = 0$ follows.

In consequence of the preceding Lemma 2 the following is obtained:

$$\begin{aligned}
S\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L &= \langle S, 2\nabla^L_{\operatorname{grad} \varrho} \operatorname{grad} \varrho\rangle_L\\
&= 2\{\operatorname{grad} \varrho \langle S, \operatorname{grad} \varrho\rangle_L - \langle \nabla^L_{\operatorname{grad} \varrho} S, \operatorname{grad} \varrho\rangle_L\} = -2\langle \nabla^L_{\operatorname{grad} \varrho} S, \operatorname{grad} \varrho\rangle_L\\
&= 2\langle \mathcal{L}_S \operatorname{grad} \varrho - \nabla^L_S \operatorname{grad} \varrho\rangle_L = 2\langle A_S \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = 0
\end{aligned}$$

by above already established equalities and by the fact that since S is a Killing field of $(L, \langle, \rangle_L)$, the endomorphism $A_S = \mathcal{L}_S - \nabla^L_S$ is skew-symmetric.

According to Lemma 3 below the level sets of ϱ are diffeomorphic to $\mathbb{R}$. Furthermore $S\varrho = 0$ implies that S is tangent to level lines of ϱ. But then $S\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = 0$ implies that a level set of $\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L$ is obtainable as a union of level lines of ϱ. Therefore the existence of a smooth function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle = \phi \circ \varrho$$

follows by the implicit function theorem.

Since $S\Delta\varrho = 0$ is valid by a preceding calculation, an obvious similar argument yields that

$$\Delta \varrho = \psi \circ \varrho$$

holds with a smooth function $\psi : \mathbb{R}^+ \to \mathbb{R}$.

In order to show that the conditions of the theorem are sufficient as well assume now that functions $\phi,\ \psi$ satisfying those conditions exist. Then there is a Killing field $S : L \to TL$ of the 2-dimensional Lorentz manifold $(L, \langle, \rangle_L)$ according to Theorem 3 below. Moreover, the canonical lift $\tilde{S} \in \mathcal{L}(L)$ is a Birkhoff field by Proposition 1. But then $\tilde{S}$ yields a Birkhoff field X^* on M^* by the equivariant isometry $\Xi : M^* \to L \times_\varrho \mathbb{S}^2$. Since M is a normal spherically symmetric space–time, either $M = M^*$ or $M = A \cup M^*$ holds where A is an axis [17]. But then X^* extends uniquely to a Killing field X of M which is obviously a Birkhoff field as well. □

4. The existence of Birkhoff fields and the Ricci tensor

Those results concerning the generalization of Birkhoff's theorem which were obtained up to now were achieved by imposing some conditions on the Einstein tensor. It seems therefore justified to discuss the question of what bearings those necessary and sufficient conditions which were presented here above have on the Einstein tensor. Considering that the Einstein gravitational tensor G of a space–time is given by the Ricci form r and the scalar curvature S as follows:

$$G = r - \frac{1}{2}Sg,$$

the above problem can be formulated in terms of the Ricci form r alone.

Let $(M, \langle, \rangle)$ be a normal spherically symmetric space–time, $z \in M^*$ and $L \subset M^*$ the leaf passing through z; then the orthogonal direct sum decomposition

$$T_z M = T_z L \overset{\perp}{\oplus} T_z G(z)$$

holds. Consider now the Ricci form $r_z : T_z M \times T_z M \to \mathbb{R}$ and the Ricci endomorphism $R_z : T_z M \to T_z M$ of M; then the equality

$$r_z(u, v) = \langle R_z u, v\rangle = 0, \quad u \in T_z L,\ v \in T_z G(z),$$

holds by an expression for the Ricci form of warped products ([11], pp. 210–211), since M^* is canonically identified with $Q = L \times_\varrho \mathbb{S}^2$ as before. Consequently, the inclusions

$$R_z(T_zL) \subset T_zL, \qquad R_z(T_zG(z)) \subset T_zG(z)$$

hold. Therefore, if $s \in T_zM$ is an eigenvector of the Ricci endomorphism R_z and

$$s = s_L + s_{G(z)}, \quad s_L \in T_zL,\ s_{G(z)} \in T_zG(z)$$

its corresponding decomposition, then $s_L,\ s_{G(z)}$ are obviously eigenvectors of R_z as well, with eigenvalues equal to that of s. Consider now the restriction

$$R_z \lceil T_zG(z) : T_zG(z) \to T_zG(z),$$

of the Ricci endomorphism. Since $T_zG(z)$ is a spacelike subspace, the restricted Ricci endomorphism being symmetric has an eigenvector. But as R_z is equivariant with respect to the linear isotropy representation which now acts transitively on the set of unit vectors in $T_zG(z)$, each non-zero vector in $T_zG(z)$ is an eigenvector of R_z and they have the same eigenvalue λ_z.

Definition 2. The non-zero vectors $v \in T_zG(z)$ will be called the *trivial eigenvectors* and their common eigenvalue λ_z the *trivial eigenvalue* of the Ricci endomorphism.

Assume as above that the spherically symmetric space–time is normal, fix a leaf $L \subset M^*$, and identify the principal part M^* with the warped product

$$Q = L \times_\varrho \mathbb{S}^2.$$

Let now r^L be the Ricci form of L and $\widetilde{r^L}$ its pull back by π_L. If $X, Y \in \mathcal{T}(Q)$ are horizontal valued then

$$r(X, Y) = \widetilde{r^L}(X, Y) - \frac{2}{\varrho \circ \pi_L}\widetilde{h_\varrho}(X, Y)$$

holds where $\widetilde{h_\varrho}$ is the pull back of the Hesse form h_ϱ of ϱ by π_L ([11], pp. 210–211). But for Ricci form r^L and the sectional curvature κ^L of L the following holds:

$$r^L(X'Y') = \kappa^L\langle X', Y'\rangle_L, \quad X', Y' \in \mathcal{T}(L).$$

Consequently, for the Ricci endomorphism R of Q, for the lift $\widetilde{R^L}$ of the Ricci endomorphism R^L of L and for the lift $\widetilde{H_\varrho}$ of the Hesse endomorphism H_ϱ associated with h_ϱ (for the lift of these endomorphisms see e.g. [11], p. 21) the following holds:

$$\left\langle RX, Y\rangle = \langle \kappa^L \circ \pi_L \cdot X - \frac{2}{\varrho \circ \pi_L}\widetilde{H_\varrho}X, Y\right\rangle,$$

where X, Y are horizontal valued. Furthermore, if $U, V \in \mathcal{T}(Q)$ are vertical valued then the following holds:

$$r(U, V) = \widetilde{r^{\mathbb{S}^2}}(U, V) - \langle U, V\rangle \left(\frac{\Delta\varrho}{\varrho} + \frac{\langle \operatorname{grad}\varrho, \operatorname{grad}\varrho\rangle_L}{\varrho^2}\right) \circ \pi_L$$

where $\widetilde{r^{\mathbb{S}^2}}$ is the lift of the Ricci form of $\mathbb{S}^2$ by $\pi_{\mathbb{S}^2}$ and $\Delta\varrho$ is the Laplacian ([11], pp. 210–211). Considering the expression for the Ricci form $r^{\mathbb{S}^2}$ by the sectional curvature $\kappa^{\mathbb{S}^2} = 1$ of the unit sphere, the following is obtained for the Ricci endomorphism R of Q and for the lift of the Ricci endomorphism of $\mathbb{S}^2$ by $\pi_{\mathbb{S}^2}$:

$$\langle RU, V\rangle = \left\langle \left(1 - \left(\frac{\Delta\varrho}{\varrho} - \frac{\langle \operatorname{grad}\varrho, \operatorname{grad}\varrho\rangle_L}{\varrho^2}\right) \circ \pi_{\mathbb{S}^2}\right) U, V\right\rangle,$$

where U, V are vertical valued. Subsequently the smooth vector field $Z : L \to TL$ will be also applied which is defined as follows:

$$\langle \operatorname{grad}\varrho(z), Z(z)\rangle_L = 0, \quad \|Z(z)\| = 1$$

for $z \in L$ and the ordering $(\operatorname{grad}\varrho(z), Z(z))$ defines the same orientation of L for each $z \in L$.

The following proposition shows now what restrictions the existence of a Birkhoff field imposes on the Ricci endomorphism of a spherically symmetric space–time.

Proposition 2. *Let $(M, \langle,\rangle)$ be a normal spherically symmetric space–time, $L \subset M^*$ a leaf, $\varrho = \rho \lceil L$ the restricted radial function and $\varrho(L) = (\alpha, \omega)$, $0 \le \alpha < \omega \le \infty$. Then the following hold:*

(1) *There is a smooth function $\phi : (\alpha, \omega) \to \mathbb{R}$ such that*

$$\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L = \phi \circ \varrho$$

holds if and only if $\operatorname{grad} \varrho(z) \in T_z L$ is an eigenvector of the Ricci endomorphism R^L_z for $z \in L$.

(2) *If $\operatorname{grad} \varrho(z)$ is an eigenvector of the Ricci endomorphism R^L_z then it is also eigenvector of the Hesse endomorphism $H\varrho_z$ with the eigenvalue*

$$\frac{1}{2}\epsilon(\operatorname{grad} \varrho) \cdot (\phi' \circ \varrho) \cdot (\phi \circ \varrho)(z), \quad z \in L.$$

(3) *Let $\operatorname{grad} \varrho(z)$ be an eigenvector of the Hesse endomorphism $H\varrho_z$ at each $z \in L$. Then there is a smooth function $\psi : (\alpha, \omega) \to \mathbb{R}$ such that*

$$\Delta \varrho = \psi \circ \varrho$$

is valid, if and only if the eigenvalue of the eigenvector $Z(z)$ of the Hesse endomorphism $H\varrho_z$ is a function of ϱ.

Proof. (1) Observe first that in consequence of an expression for the Ricci endomorphism R above, $\widetilde{\operatorname{grad} \varrho}$ yields an eigenvector of R if and only if it is an eigenvector of the lift $\widetilde{H\varrho}$ of the Hesse endomorphism associated with ϱ.

Assume first that there is a smooth function ϕ such that

$$\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle_L = \phi \circ \varrho$$

holds. Then with the above already defined smooth vector field $Z : L \to TL$ the following holds:

$$\begin{aligned}\langle H\varrho(\operatorname{grad} \varrho), Z\rangle_L &= \langle H\varrho(Z), \operatorname{grad} \varrho\rangle_L = \langle \nabla^L_Z \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L \\ &= \frac{1}{2} Z\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = \frac{1}{2} Z(\phi \circ \varrho) = \frac{1}{2}(\phi' \circ \varrho)\langle Z, \operatorname{grad} \varrho\rangle = 0.\end{aligned}$$

Therefore $\operatorname{grad} \varrho$ is an eigenvector of the Hesse endomorphism $H\varrho$ associated with ϱ. But then $\widetilde{\operatorname{grad} \varrho}$ is an eigenvector of the Ricci endomorphism R_z as well.

Assume conversely that $\widetilde{\operatorname{grad} \varrho}$ is an eigenvector of the Ricci endomorphism. Then by the preceding calculation

$$Z\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = 0$$

holds and therefore any level set of $\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle$ is a union of level lines of ϱ. But from this fact the existence of the smooth function ϕ follows as was shown already above.

(2) Assume now that $\widetilde{\operatorname{grad} \varrho}$ is an eigenvector of the Ricci endomorphism; then $\operatorname{grad} \varrho$ is an eigenvector of the Hesse endomorphism and its corresponding eigenvalue is given by

$$\begin{aligned}\lambda_{\operatorname{grad} \varrho} &= \left\langle H\varrho \frac{\operatorname{grad} \varrho}{\|\operatorname{grad} \varrho\|}, \frac{\operatorname{grad} \varrho}{\|\operatorname{grad} \varrho\|}\right\rangle_L = \frac{1}{\|\operatorname{grad} \varrho\|^2}\langle \nabla_{\operatorname{grad} \varrho} \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L \\ &= \frac{1}{\|\operatorname{grad} \varrho\|^2}\frac{1}{2}\operatorname{grad} \varrho\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = \frac{1}{2\|\operatorname{grad} \varrho\|^2}(\operatorname{grad} \varrho)(\phi \circ \varrho) \\ &= \frac{1}{2\|\operatorname{grad} \varrho\|^2} \cdot (\phi' \circ \varrho) \cdot \langle \operatorname{grad} \varrho, \operatorname{grad} \varrho\rangle_L = \frac{1}{2}\epsilon(\operatorname{grad} \varrho) \cdot (\phi' \circ \varrho) \cdot (\phi \circ \varrho),\end{aligned}$$

where $\|\operatorname{grad} \varrho\| \neq 0$ was assumed as above, but $\epsilon(\operatorname{grad} \varrho) = \pm 1$ is admitted.

(3) Since $\Delta \varrho$ can be expressed by means of the eigenvalues of the Hesse endomorphism $H\varrho$ ([11], pp. 86–87), the Laplacian $\Delta \varrho$ is a function of ϱ if and only if the eigenvalue of the eigenvector Z of $H\varrho$ is a function of ϱ under the given assumptions. □

The earlier results of Cahen and Debever [4], Goenner [7], and of Barnes [1] on generalization of Birkhoff's theorem were summarized in terms of the Segré symbol ([10], pp. 157–158). The Segré symbol of a Ricci endomorphism is a sequence of natural numbers which give the multiplicities of the elementary divisors of the endomorphism displayed in such an order that the multiplicities of those elementary divisors which correspond to eigenvalues of spacelike eigenvectors precede those which correspond to eigenvalues of lightlike and timelike ones and are separated by a comma from them; moreover, multiplicities of elementary divisors which correspond to the same eigenvalue are enclosed in brackets ([10], pp. 66–89). A concise account of the earlier generalizations of Birkhoff's theorem is now as follows ([10], pp. 157–158):

Theorem 2 (*Cahen and Debever, Goenner, Barnes*). *Let* $(M, \langle,\rangle)$ *be a space–time where an isometric action of a 3-dimensional Lie group with non-lightlike orbits of maximal dimension* 2 *is given. If the Segré symbol of Ricci tensor of the space–time is* $[(111, 1)]$ *or* $[(11)(1, 1)]$ *then the action extends to an isometric action of a 4-dimensional Lie group with non-lightlike orbits of maximal dimension* 3.

The Segré symbol $[(111, 1)]$ corresponds to a Ricci tensor with one eigenvalue of multiplicity 4; this is the case of an Einstein manifold, while $[(11)(1, 1)]$ corresponds to a Ricci tensor with two eigenvalues of multiplicity 2 each. In the case of a spherically symmetric space–time with Segré symbol $[(111, 1)]$ or $[(11)(1, 1)]$ the first two elements of the symbol obviously correspond to the trivial eigenvalue of eigenvectors in $T_z G(z)$, while the third and the fourth element correspond to an eigenvalue of eigenvectors in $T_z L$.

Proposition 3. *Let* $(M, \langle,\rangle)$ *be a normal spherically symmetric space–time such that for* $z \in M^*$ *the Ricci tensor has the Segré symbol* $[(111, 1)]$ *or* $[(11)(1, 1)]$. *Then the space–time admits a Birkhoff field.*

Proof. In fact, the given values of the Segré symbol and preceding observations yield that there are at least two eigenvectors of the Ricci endomorphism in $T_z L$ having the same eigenvalue. But then each non-zero vector in $T_z L$ is an eigenvector of the Ricci endomorphism; in particular $\operatorname{grad} \varrho$ yields an eigenvector as well. Therefore the preceding Proposition 2 applies and the existence of the required function ϕ follows. Moreover, since $\Delta\varrho$ can be expressed in terms of the eigenvalues of the Hesse endomorphism $H\varrho$, and these eigenvalues are equal, and, furthermore, their common value is a function of ϱ by the preceding Proposition 2, the existence of the required function ψ also follows. □

5. A construction of spherically symmetric space–times admitting a Birkhoff field

Some spherically symmetric space–times are constructed below which admit a Birkhoff field but do not satisfy those above presented conditions which were used in earlier generalizations of Birkhoff's theorem. The construction is based on the sufficient conditions given in Proposition 1 already here.

Consider the 1-dimensional Lorentz manifold $\mathbb{R}^1_1$ obtained from $\mathbb{R}^1$ by multiplying its metric with -1 and consider a smooth function

$$\mu : \mathbb{R}^1_1 \to \mathbb{R}^+.$$

Then the Lorentzian warped product

$$L = \mathbb{R}^1_1 \times_\mu \mathbb{R}^1$$

is a 2-dimensional simply connected non-compact Lorentz manifold. There is a canonical global coordinate system (u, v) on L defined by its product structure. For the corresponding base fields

$$\langle\partial_u, \partial_u\rangle_L = -1, \qquad \langle\partial_u, \partial_v\rangle_L = 0, \qquad \langle\partial_v, \partial_v\rangle_L = \mu^2$$

holds. Then the timelike vector field ∂_u defines a time orientation of L; moreover, ∂_v is obviously a Killing field on L. Let now

$$\varrho : L \to \mathbb{R}^+$$

be a smooth function such that $\partial_v \varrho = 0$ holds; then there is a smooth function $\widehat{\varrho} : \mathbb{R} \to \mathbb{R}$ such that

$$\varrho(z) = \widehat{\varrho} \circ u(z), \quad z \in L$$

holds, and conversely. Now the warped product

$$Q = L \times_{\varrho} \mathbb{S}^2$$

yields a spherically symmetric space–time with the time orientation induced by that of L and Q admits a Birkhoff field by Proposition 1 above.

In order to see that the above space–time Q does not satisfy those conditions which were used in earlier generalizations of Birkhoff's theorem, the Ricci endomorphism of Q will be calculated subsequently.

An obvious calculation yields that

$$\operatorname{grad} \varrho = \frac{\partial \widehat{\varrho}}{\partial u} \partial_u$$

holds, and accordingly it is also assumed that $\frac{\partial \widehat{\varrho}}{\partial u}$ is nowhere 0 on L.

Then for the Hessian form h_{ϱ} the following is obtained:

$$\begin{aligned} h_{\varrho}(\partial_u, \partial_u) &= \langle \nabla^L_{\partial_u} \operatorname{grad} \varrho, \partial_u \rangle_L = \partial_u \langle \operatorname{grad} \varrho, \partial_u \rangle_L - \langle \operatorname{grad} \varrho, \nabla^L_{\partial_u} \partial_u \rangle_L \\ &= -\partial_u \frac{\partial \widehat{\varrho}}{\partial u} = \frac{\partial^2 \widehat{\varrho}}{\partial u^2}, \end{aligned}$$

and, in fact, the field ∂_u has unit norm and therefore $\nabla_{\partial_u} \partial_u \perp \partial_u,\ \operatorname{grad} \varrho$ is valid. Moreover,

$$\begin{aligned} h_{\varrho}(\partial_u, \partial_v) &= \langle \nabla^L_{\partial_u} \operatorname{grad} \varrho, \partial_v \rangle_L = \partial_u \langle \operatorname{grad} \varrho, \partial_v \rangle_L - \langle \operatorname{grad} \varrho, \nabla^L_{\partial_u} \partial_v \rangle_L \\ &= -\frac{\partial \varrho}{\partial u} \langle \partial_u, \nabla^L_{\partial_u} \partial_v \rangle_L = -\frac{\partial \widehat{\varrho}}{\partial u} \langle \partial_u, \nabla^L_{\partial_v} \partial_u \rangle_L = 0 \end{aligned}$$

is valid too. Thus $\partial_U,\ \partial_v$ are eigenvectors of the Hesse endomorphism H_{ϱ}. Furthermore,

$$\begin{aligned} h_{\varrho}(\partial_v, \partial_v) &= \langle \nabla_{\partial_v} \operatorname{grad} \varrho, \partial_v \rangle_L = \partial_v \langle \operatorname{grad} \varrho, \partial_v \rangle_L - \langle \operatorname{grad} \varrho, \nabla_{\partial_v} \partial_v \rangle_L \\ &= -\frac{\partial \widehat{\varrho}}{\partial u} \langle \partial_u, \nabla_{\partial_v} \partial_v \rangle_L = -\frac{\partial \widehat{\varrho}}{\partial u} \left(\partial_v \langle \partial_u, \partial_v \rangle_L - \langle \nabla_{\partial v} \partial_u, \partial_v \rangle_L \right) \\ &= -\frac{\partial \widehat{\varrho}}{\partial u} \left(-\langle \nabla_{\partial_u} \partial_v, \partial_v \rangle_L \right) = \frac{\partial \widehat{\varrho}}{\partial u} \frac{1}{2} \partial_u \langle \partial_v, \partial_v \rangle_L \\ &= \frac{\partial \widehat{\varrho}}{\partial u} \frac{1}{2} \partial_u (\mu^2) = \frac{\partial \widehat{\varrho}}{\partial u} \frac{\partial \mu}{\partial u} \mu. \end{aligned}$$

holds. The Laplacian Δ_{ϱ} can be also obtained now by the fact that it is the divergence of the field $\operatorname{grad} \varrho$ and therefore it is obtainable as the sum of the values of the Hessian form h_{ϱ} on an orthonormal system ([11], pp. 86–87):

$$\begin{aligned} \Delta_{\varrho} &= \epsilon(\partial_u) \langle \nabla^L_{\partial_u} \operatorname{grad} \varrho, \partial_u \rangle_L + \frac{1}{\|\partial_v\|^2} \epsilon(\partial_v) \langle \nabla^L_{\partial_v} \operatorname{grad} \varrho, \partial_v \rangle_L \\ &= (-1) \frac{\partial^2 \widehat{\varrho}}{\partial u^2} + \frac{1}{\mu^2} \frac{\partial \widehat{\varrho}}{\partial u} \frac{\partial \mu}{\partial u} \mu. \end{aligned}$$

The sectional curvature κ^L of L is obtained by application of a general formula of the sectional curvature of a 2-dimensional semi-Riemann manifold in a coordinate system with orthogonal coordinate lines ([11], pp. 80–81). In fact, put

$$\begin{aligned} & E = \langle \partial_u, \partial_u \rangle_L = -1, \qquad G = \langle \partial_v, \partial_v \rangle_L = \mu^2, \\ & e = |E|^{1/2} = 1, \qquad g = |G|^{1/2} = \mu, \qquad \epsilon_1 = -1, \qquad \epsilon_2 = 1. \\ & \kappa^L = \frac{-1}{eg} \left(\epsilon_1 \left(\frac{g_u}{e} \right)_u + \epsilon_2 \left(\frac{e_v}{g} \right)_v \right) \\ & \quad = \frac{-1}{1 \cdot \mu} \left((-1) \left(\frac{\frac{\partial \mu}{\partial u}}{1} \right)_u + 1 \left(\frac{0}{\mu} \right)_v \right) = \frac{1}{\mu} \frac{\partial^2 \mu}{\partial u^2}. \end{aligned}$$

The eigenvalues of the Ricci endomorphism R of the above space–time $M = L \times_{\varrho} \mathbb{S}^2$ are the following:

$$\frac{1}{\mu}\frac{\partial^2 \mu}{\partial u^2} + \frac{\partial^2 \widehat{\varrho}}{\partial u^2},$$
$$\frac{1}{\mu}\frac{\partial^2 \mu}{\partial u^2} + \frac{\partial \widehat{\varrho}}{\partial u}\frac{\partial \mu}{\partial u}\mu,$$
$$1 - \frac{1}{\varrho}\left(-\frac{\partial^2 \widehat{\varrho}}{\partial u^2} + \frac{1}{\mu}\frac{\partial \widehat{\varrho}}{\partial u}\frac{\partial \mu}{\partial u}\right) - \frac{1}{\varrho^2}\left(\frac{\partial \widehat{\varrho}}{\partial u}\right)^2.$$

As the above expressions of the eigenvalues of the Ricci endomorphism show, the validity of the conditions for the existence of a Birkhoff field given earlier are not fulfilled in general in the case of those spherically symmetric space–times which were constructed here.

Appendix. A solution of the global Lorentzian Minding problem

The solution of the Minding problem of classical surface theory yields a sufficient condition for the existence of a Killing field on a surface in terms of a function which has to be invariant under the Killing field; such a function can be the Gaussian curvature considering the *theorema egregium* (see e.g. [19], vol. II, pp. 85–87, 220–228). The analogous global version of the Minding problem on a 2-dimensional Lorentz manifold is solved in what follows.

Lemma 3. *Let $\varrho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $d\varrho$ is nowhere 0 on $\mathbb{R}^2$. Then the level sets of ϱ are diffeomorphic to $\mathbb{R}$.*

Proof. If N is a level set of ϱ then it is locally diffeomorphic to $\mathbb{R}$ since $d\varrho \neq 0$ everywhere on $\mathbb{R}^2$. Therefore a connected component C of N is diffeomorphic either to $\mathbb{R}$ or to the circle $\mathbb{S}^1$. Assume that C is diffeomorphic to $\mathbb{S}^1$; then $C = \partial D$, where $D \subset \mathbb{R}^2$ is a disk. But then ϱ has an extremal value in D and there $d\varrho = 0$, a contradiction.

Assume that $C,\ C' \subset N$ are two connected components of N, and consider a smooth curve $\varphi : [0, 1] \to \mathbb{R}^2$ with $\varphi(0) \in C,\ \varphi(1) \in C'$. There is no loss of generality in assuming that each $\tau \in [0, 1]$ has a neighbourhood where φ intersects a level set of ϱ at most once. But there is a $\tau_0 \in (0, 1)$ where $\varrho \circ \varphi$ has an extremal value and therefore there are $\tau' < \tau_0 < \tau''$ arbitrary near to τ_0 with $\varrho \circ \varphi(\tau') = \varrho \circ \varphi(\tau'')$, a contradiction. □

Theorem 3. *Let $(L, \langle,\rangle)$ be a 2-dimensional simply connected non-compact Lorentz manifold and $\varrho : L \to \mathbb{R}$ a smooth function such that $\|\mathrm{grad}\,\varrho\|$ is nowhere 0 on L; put $\varrho(L) = (\alpha, \omega)$ where $0 \leq \alpha < \omega \leq \infty$. Then the following assertions are valid:*

(1) *Assume that there is a smooth function $\phi : (\alpha, \omega) \to \mathbb{R}$ such that*

$$\langle \mathrm{grad}\,\varrho, \mathrm{grad}\,\varrho \rangle = \phi \circ \varrho$$

holds. Then the integral curves of grad ϱ *are pregeodesics. Furthermore, there is a smooth function $\varsigma : L \to \mathbb{R}$ such that*

$$L \ni z \mapsto (\varrho(z), \varsigma(z)) \in (\alpha, \omega) \times \mathbb{R} \subset \mathbb{R}^2$$

is a smooth coordinate system on L with orthogonal coordinate lines.

(2) *Assume that there is a smooth function $\psi : (\alpha, \omega) \to \mathbb{R}$ such that*

$$\Delta\varrho = \psi \circ \varrho$$

holds for the Laplacian of ϱ. Then there is a Killing field $S : L \to TL$ of the Lorentz manifold L such that $S\varrho = 0$ is valid everywhere on L.

Proof. (1) Assume the existence of the function $\phi : (\alpha, \omega) \to \mathbb{R}$ with the above given property. Then by the former Lemma 2

$$\begin{aligned} 2\nabla_{\mathrm{grad}\,\varrho}\mathrm{grad}\,\varrho &= \mathrm{grad}\,\langle \mathrm{grad}\,\varrho, \mathrm{grad}\,\varrho \rangle \\ &= \mathrm{grad}\,(\phi \circ \varrho) = (\phi' \circ \varrho) \cdot \mathrm{grad}\,\varrho \end{aligned}$$

holds which means that the integral curves of grad ϱ are pregeodesics.

Consider now the smooth vector field $R : L \to TL$ defined as follows:

$$R : L \ni z \mapsto \frac{\operatorname{grad} \varrho}{\langle \operatorname{grad} \varrho, \operatorname{grad} \varrho \rangle}$$

and also the maximal local 1-parameter group of diffeomorphisms generated by R, which is a smooth map

$$\Gamma : U \to L$$

where $U \subset \mathbb{R} \times L$ is a neighbourhood of the set $\{0\} \times L$ such that

$$U \cap (\mathbb{R} \times \{z\}) = (\xi_z, \eta_z) \times \{z\}$$

holds for $z \in L$ with $-\infty \leq \xi_z < \eta_z \leq \infty$, and that

$$\gamma_z = \Gamma \lceil ((\xi_z, \eta_z) \times \{z\})$$

is the maximal integral curve of R with $\gamma_z(0) = z$ (see e.g. [6], pp. 99–104).

Consider now the smooth function $\varrho \circ \gamma_z : (\xi_z, \eta_z) \to (\alpha, \omega)$ for $z \in L$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\varrho \circ \gamma_z) = \dot{\gamma}_z \varrho = \langle \dot{\gamma}_z, \operatorname{grad} \varrho \rangle = \langle R, \operatorname{grad} \varrho \rangle = 1$$

holds. Therefore

$$\alpha < \varrho \circ \gamma_z(\tau) = \varrho(z) + \tau < \omega$$

is valid. Even the equalities

$$\alpha = \varrho(z) + \xi_z < \varrho(z) + \eta_z = \omega$$

hold, considering that the integral curve γ_z is maximal. Fix now a $\theta \in (\alpha, \omega)$; then by the preceding Lemma 3 the 1-dimensional submanifold

$$A = \varrho^{-1}(\theta)$$

is diffeomorphic to $\mathbb{R}$ and therefore there is a smooth diffeomorphism

$$\varsigma^0 : A \to \mathbb{R}$$

which can be reparametrized so as to be an isometry or an anti-isometry depending on whether A is spacelike or timelike. Then, by basic properties of integral curves of smooth vector fields, any point $z \in L$ is obtainable as

$$z = \Gamma(\tau, x)$$

where $\tau \in (\alpha - \theta, \omega - \theta)$ and $x \in A$ are unique and depend smoothly on z. Thus a smooth function

$$\varsigma : L \to \mathbb{R}$$

is defined by $\varsigma(z) = \varsigma^0(x)$. Consequently a smooth coordinate system is obtained on L by

$$L \ni z \mapsto (\varrho(z), \varsigma(z)) \in (\alpha, \omega) \times \mathbb{R},$$

where the coordinate lines are obviously orthogonal.

(2) Assume now the existence of a smooth function $\psi : (\alpha, \omega) \to \mathbb{R}$ with the given property. Consider the smooth vector field

$$E : L \ni z \mapsto \frac{\operatorname{grad} \varrho}{\|\operatorname{grad} \varrho\|}.$$

Then $E(z)$ is an eigenvector of the Hesse endomorphism $H\varrho_z : T_z L \to T_z L$ for $z \in L$. Namely, there is a unique smooth vector field $N : L \to TL$ such that

$$\|N(z)\| = 1, \qquad \langle N(z), E(z) \rangle = 0$$

holds for $z \in L$ and the ordering $(E(z), N(z))$ defines the same orientation of L for every $z \in L$. Then

$$\begin{aligned}\langle H\varrho E, N\rangle &= h\varrho(E, N) = \langle \nabla_E \operatorname{grad}\varrho, N\rangle \\ &= E\langle \operatorname{grad}\varrho, N\rangle - \langle \operatorname{grad}\varrho, \nabla_E N\rangle = 0\end{aligned}$$

is valid; in fact, the integral curves of E are geodesics and N is parallel along these curves. In order to calculate the eigenvalue of the eigenvector E observe that

$$\begin{aligned}\langle H\varrho E, E\rangle &= h\varrho(E, E) = \langle \nabla_E \operatorname{grad}\varrho, E\rangle \\ &= E\langle \operatorname{grad}\varrho, E\rangle - \langle \operatorname{grad}\varrho, \nabla_E E\rangle \\ &= E\left(\frac{1}{\|\operatorname{grad}\varrho\|}\langle \operatorname{grad}\varrho, \operatorname{grad}\varrho\rangle\right) = E\left(\frac{1}{|\phi\circ\varrho|^{1/2}}\cdot(\phi\circ\varrho)\right) \\ &= E(\epsilon(\operatorname{grad}\varrho)\cdot(\phi\circ\varrho)^{1/2}) = \epsilon(\operatorname{grad}\varrho)\cdot\frac{1}{2}(\phi\circ\varrho)^{-1/2}\cdot(\phi'\circ\varrho)\cdot\langle E, \operatorname{grad}\varrho\rangle \\ &= \frac{1}{2}\epsilon(\operatorname{grad}\varrho)(\phi\circ\varrho)^{-1/2}\cdot(\phi'\circ\varrho)\cdot\epsilon(\operatorname{grad}\varrho)(\phi\circ\varrho)^{1/2} \\ &= \frac{1}{2}\phi'\circ\varrho.\end{aligned}$$

The above equality yields that the eigenvalue of the Hesse endomorphism $H\varrho$ corresponding to E is a function of ϱ. But the facts that $\Delta\varrho$ is a function of ϱ and that $\Delta\varrho$ is obtainable as sum of eigenvalues of the Hesse endomorphism $H\varrho$ imply that the eigenvalue of $H\varrho$ corresponding to the eigenvector N is given by $\lambda_N \circ \varrho$ as a function of ϱ as well. Consequently,

$$\begin{aligned}0 &= N\langle N, \operatorname{grad}\varrho\rangle = \langle \nabla_N N, \operatorname{grad}\varrho\rangle + \langle N, \nabla_N \operatorname{grad}\varrho\rangle \\ &= \langle \nabla_N N, \operatorname{grad}\varrho\rangle + \langle H\varrho N, N\rangle\end{aligned}$$

yields that $\langle \nabla_N N, \operatorname{grad}\varrho\rangle$ too is a function of ϱ.

It will be shown now that $\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle = 0$ holds everywhere on L. In fact

$$\begin{aligned}\partial_\varrho\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle &= \partial_\varsigma\partial_\varrho\langle\partial_\varsigma, \partial_\varsigma\rangle = 2\partial_\varsigma\langle\nabla_{\partial_\varrho}\partial_\varsigma, \partial_\varsigma\rangle = 2\partial_\varsigma\langle\nabla_{\partial_\varsigma}\partial_\varrho, \partial_\varsigma\rangle \\ &= 2\partial_\varsigma\left\langle\nabla_{\partial_\varsigma}\left(\frac{1}{\phi\circ\varrho}\operatorname{grad}\varrho\right), \partial_\varsigma\right\rangle = 2\partial_\varsigma\left(\frac{1}{\phi\circ\varrho}\langle\nabla_{\partial_\varsigma}\operatorname{grad}\varrho, \partial_\varsigma\rangle\right) \\ &= 2\frac{1}{\phi\circ\varrho}\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle\langle\nabla_N \operatorname{grad}\varrho, N\rangle = \frac{2\lambda_N}{\phi}\circ\varrho\cdot\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle.\end{aligned}$$

But by the preceding calculations the following must be valid on L:

$$\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle = (\xi\circ\varsigma)\cdot(\mathrm{e}^{\eta\circ\varrho}),$$

where $\xi : \mathbb{R} \to \mathbb{R}$ is some smooth function and $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$\eta' = \frac{2\lambda_N}{\phi}$$

is valid. In fact,

$$\begin{aligned}\partial_\varrho\left((\xi\circ\varrho)\cdot(\mathrm{e}^{\eta\circ\varrho})\right) &= (\xi\circ\varrho)\cdot\mathrm{e}^{\eta\circ\varrho}\cdot(\eta'\circ\varrho)\langle\partial_\varrho, \operatorname{grad}\varrho\rangle \\ &= (\xi\circ\varrho)\cdot\mathrm{e}^{\eta\circ\varrho}\cdot(\eta'\circ\varrho)\end{aligned}$$

holds. Considering the fact that

$$(\partial_\varsigma\langle\partial_\varsigma, \partial_\varsigma\rangle)\lceil A = 0,$$

is valid and that the exponential function is nowhere 0, the function ξ has to be identically zero. But this implies the assertion.

In order to show that ∂_ς is a Killing field put now $\langle,\rangle = g$ for convenience. Since $\mathcal{L}_{\partial_\varsigma} g$ is a tensor, it is enough to show that the equalities

$$(\mathcal{L}_{\partial_\varsigma})(\partial_\varrho, \partial_\varrho) = (\mathcal{L}_{\partial_\varsigma})(\partial_\varrho, \partial_\varsigma) = (\mathcal{L}_{\partial_\varsigma} g)(\partial_\varsigma, \partial_\varsigma) = 0$$

are valid. In fact, the following equalities are valid by the preceding observations:

$$\left(\mathcal{L}_{\partial_\varsigma} g\right)(\partial_\varrho, \partial_\varrho) = \partial_\varsigma \left(g(\partial_\varrho, \partial_\varrho)\right) = \partial_\varsigma \left(\frac{1}{\phi \circ \varrho}\right) = 0,$$
$$\left(\mathcal{L}_{\partial_\varsigma} g\right)(\partial_\varrho, \partial_\varsigma) = \partial_\varsigma (g(\partial_\varrho, \partial_\varsigma)) = 0,$$
$$\left(\mathcal{L}_{\partial_\varsigma} g\right)(\partial_\varsigma, \partial_\varsigma) = \partial_\varsigma (g(\partial_\varsigma, \partial_\varsigma)) = 0. \quad \square$$

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